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HS 2018

Lineare Algebra für die Naturwissenschaften
Probeprüfung

Regeln

- Die Antworten auf die Fragen müssen klar, verständlich und begründet sein. Unlesbare Lösungen werden nicht korrigiert.
- Die Prüfung dauert zwei Stunden.
- Es sind keine Notizen erlaubt.
- Die Benutzung von elektronischen Hilfsmitteln ist verboten.
- Bitte verwenden Sie keine eigenen Blätter. Melden Sie sich, falls Sie mehr Papier brauchen.
- Schreiben Sie Ihre Lösungen unter die Aufgaben. Sie können auch die Rückseite verwenden.
- Schreiben Sie Namen und Matrikelnummer auf jedes Blatt.
- 4 richtig gelöste Aufgabe ergeben die maximale Note.
- Jede Aufgabe gibt die gleiche Anzahl Punkte.

Viel Erfolg!

Exercise	1	2	3	4	5	total
Points						

Name:

Matrikelnummer:

Exercise 1

Decide whether the statements below are true or false. Correct answers count +0.5 points, wrong answers count as -0.5points. If the total for this question is negative it will be counted as 0 points towards the total.

(a) The matrix $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ is orthogonal.

☐ True ☒ False

(columns are not normalized)

(b) The equation $z^4 = 1$ has a unique solution in $\mathbb{C}$.

☐ True ☒ False

(there are 4 solutions)

(c) $(1+i)$ is an eigenvalue of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

☐ True ☒ False

(symmetric matrices have only real eigenvalues)

(d) Let A be a symmetric matrix. Then A^2 is positive semi-definite.

☒ True ☐ False

(its eigenvalues are the squares of the eigenvalues of A, hence at least 0)

(e) The matrix $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ is invertible.

☐ True ☒ False

(the determinant is 0)

(f) The argument of i is $\pi/2$.

☒ True ☐ False

(g) The set $W = \{f\colon \mathbb{R} \to \mathbb{R}, f(0) = 1\}$ is a subvectorspace of $V = \{f\colon \mathbb{R} \to \mathbb{R}\}$.

☐ True ☒ False

($0 \notin W$)

(h) The eigenvalues of $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ are $\cos\theta, \sin\theta$.

☐ *True* ☒ *False*

(the eigenvalues are $e^{\pm i\theta}$)

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Exercise 2 (Bases and coordinates)
Consider the three vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

(a) Show that $[v] = [v_1, v_2, v_3]$ is a basis of $\mathbb{R}^3$.

(b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map whose matrix in the standard basis is

$$T_{[e]\to[e]} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Find the representation of T with respect to the basis $[v]$.

Solution:

(a) The three vectors are linearly independent if the determinant of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

is not zero. Since $\det A = 1$ we know that v_1, v_2, v_3 form a basis.

(b) We have

$$T_{[v]\to[v]} = Id_{[e]\to[v]} T_{[e]\to[e]} Id_{[v]\to[e]}$$

and that

$$Id_{[v]\to[e]} = (v_1|v_2|v_3) = A$$
$$Id_{[e]\to[v]} = A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

hence

$$T_{[v]\to[v]} = A^{-1} T_{[e]\to[e]} A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 7 & 9 \\ -3 & -5 & -6 \end{pmatrix}$$

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Exercise 3 (Linear System)
Consider the linear system

$$\begin{cases} 3x_1 + x_2 & = 1 \\ x_2 + x_3 & = 1 \\ 2x_2 + \lambda x_3 & = 2 \end{cases}$$

depending on a parameter $\lambda \in \mathbb{R}$.

(a) For which values of λ does this system have a unique solution?

(b) Let $\lambda = 2$. Give the set of solutions of the system.

Solution:

(a) The system has a unique solution if the determinant of the coefficient matrix is non-zero, this happens for $\lambda \neq 2$.

(b) If $\lambda = 2$, the system has infinitely many solutions. If x_3 is the free parameter, then $x_2 = 1 - x_3$ and $x_1 = (1 - x_2)/3 = x_3/3$, hence the set of solutions is

$$L = \{(x_3/3, 1 - x_3, x_3) | x_3 \in \mathbb{R}\}.$$

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Exercise 4 (Diagonalization)
Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

(a) Find all eigenvalues of A and their algebraic and geometric multiplicities.

(b) Find an orthonormal basis of $\mathbb{R}^3$ of eigenvectors of A.

(c) Find an invertible matrix S such that SAS^{-1} is a diagonal matrix.

Solution:

(a) The characteristic polynomial is

$$A = \det(A - zI_3) = -z^3 + 7z^2 - 10z = -z(z-2)(z-5)$$

hence the eigenvalues are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 5$ and they all have geometric and algebraic multiplicity 1.

(b) Eigenspaces:

$$\begin{aligned} E_{\lambda_1}(A) &= \ker(A) = \{(0, x, -x)^T | x \in \mathbb{R}\} \\ E_{\lambda_2}(A) &= \ker(A - 2I_3) = \{(-2x, x, x)^T | x \in \mathbb{R}\} \\ E_{\lambda_3}(A) &= \ker(A - 5I_3) = \{(x, x, x)^T | x \in \mathbb{R}\} \end{aligned}$$

An orthonormal basis is given by normalized eigenvectors (since all eigenspaces are mutually orthogonal):

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, v_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) We have $S = (v_1|v_2|v_3)^{-1} = Id_{[e]\to[v]}$. Since the vectors form an orthonormal basis, S is orthogonal and

$$S = (v_1|v_2|v_3)^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

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Exercise 5 (ODEs)
Consider the system of ODEs

$$\begin{cases} y_1' &= 2y_1 + 3y_2 \\ y_2' &= 3y_1 + 2y_2 \end{cases}$$

(a) Give the general solution of this system of ODEs.

(b) Solve the corresponding initial value problem with $y_1(0) = 2, y_2(0) = 3$.

Solution:

We have $y' = Ay$ with $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 5$, with eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}$ hence the general solution is

$$y(t) = a_1 e^{-t} v_1 + a_2 e^{5t} v_2$$

with $a_1, a_2 \in \mathbb{R}$.

We have to solve the linear system $a_1 v_1 + a_2 v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ which yields $a_1 = -1/2, a_2 = 5/2$ hence the solution is

$$y(t) = -1/2 e^{-t} v_1 + 5/2 e^{5t} v_2 = \begin{pmatrix} -\frac{1}{2} e^{-t} + \frac{5}{2} e^{5t} \\ \frac{1}{2} e^{-t} + \frac{5}{2} e^{5t} \end{pmatrix}$$