Lineare Algebra für die Naturwissenschaften
Probeprüfung

Aufgabe 1 Betrachte den $\mathbb{R}$-Vektorraum $\mathscr{P}_3$ der Polynome in einer Variablen mit reellen Koeffizienten von Grade höchstens 3,

$$\mathscr{P}_3 = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

und bezeichne mit $[v] = [v^{(0)}, v^{(1)}, v^{(2)}, v^{(3)}]$ die Standardbasis von $\mathscr{P}_3$,

$$v^{(0)} = 1,\ v^{(1)} = t,\ v^{(2)} = t^2,\ v^{(3)} = t^3.$$

Sei $T : \mathscr{P}_3 \to \mathscr{P}_3$ die Abbildung

$$(Tp)(t) = tp'''(t) + 2p'(t) + 3p(t).$$

(i) Zeige, dass T linear ist.

///// Let's take $p, q \in \mathscr{P}_3$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}
&T(p+q)(t) = t(p+q)'''(t) + 2(p+q)'(t) + 3(p+q)(t)\\
&= tp'''(t) + tq'''(t) + 2p'(t) + 2q'(t) + 3p(t) + 3q(t) = T(p)(t) + T(q)(t)\\
&T(\alpha p)(t) = t(\alpha p)'''(t) + 2(\alpha p)'(t) + 3(\alpha p)(t)\\
&= \alpha tp'''(t) + \alpha 2p'(t) + \alpha 3p(t) = \alpha T(p)(t)
\end{aligned}$$

so the map T is linear. /////

(ii) Bestimme die Matrixdarstellung $T_{[v]\to[v]}$ von T bezüglich der Basis $[v]$.

///// We have

$$T(v^{(0)})(t) = 3, \quad T(v^{(1)})(t) = 2 + 3t, \quad T(v^{(2)})(t) = 4t + 3t^2, \quad T(v^{(3)})(t) = 6t + 6t^2 + 3t^3$$

and therefore

$$T_{[v]\to[v]} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 3 & 4 & 6 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \quad /////$$

(iii) Entscheide, ob T diagonalisierbar ist.

///// We have

$$\chi_T = (3-\lambda)^4 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 3.$$

Let us assume T would be diagonalizable, then

$$T_{[v]\to[v]} = S^{-1}\operatorname{diag}(3,3,3,3)S = 3Id_4$$

which is a contradiction! /////

Aufgabe 2

(i) Berechne die Determinante von

$$\begin{pmatrix} i & 0 & 1 \\ -1 & 1 & -1 \\ 1+i & -1+i & 1-i \end{pmatrix}^4 .$$

///// By expanding $\det(A)$ with respect to the first row one computes $\det(A) = i((1-i)+(-1+i)) + 1(1-i-1-i) = -2i$ and $\det(A^4) = \det(A)^4 = 16$. /////

(ii) Berechne den nicht orientierten Winkel zwischen den Vektoren $v = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ und $w = (\sqrt{2}, -1, 1)$ bezüglich des Euklidschen Skalarproduktes in $\mathbb{R}^3$.

/////

$$\cos(\phi) = \langle \frac{v}{||v||}, \frac{w}{||w||} \rangle = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{3}. \quad /////$$

(iii) Bestimme die Nullstellen von $p(z) = z^4 - i$.

///// The solutions of $z^4 = i = e^{i\frac{\pi}{2}}$ are given by

$$z_i = e^{i\psi_j},\ j \in \{0, \ldots, 3\} \quad \text{with} \quad \psi_j = \frac{\pi}{8} + j(\frac{\pi}{2}). \quad /////$$

Aufgabe 3

Betrachte die 3×3 Matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) Berechne die Eigenwerte von A.

///// By computing the characteristic polynomial $\chi_A = \lambda(-\lambda^2 + 1)$ we obtain the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$. /////

(ii) Bestimme eine orthonormierte Basis $[v]$ von $\mathbb{R}^3$, bestehend aus Eigenvektoren von A.

///// Let us first compute the eigenvectors. We have for $\lambda_1 = 0$

$$A - \lambda_1 Id_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

choose $\tilde{v}^{(1)} = (0, 1, 0)$ as corresponding eigenvector.

For $\lambda_2 = 1$

$$A - \lambda_2 Id_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

choose $\tilde{v}^{(2)} = (1,0,1)$ as corresponding eigenvector.

For $\lambda_3 = -1$

$$A - \lambda_3 Id_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix};$$

choose $\tilde{v}^{(3)} = (1,0,-1)$ as corresponding eigenvector. Then $v^{(1)} = (0,1,0)$, $v^{(2)} = \frac{1}{\sqrt{2}}(1,0,1)$, $v^{(3)} = \frac{1}{\sqrt{2}}(1,0,-1)$ is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A. /////

(iii) Berechne A^{20}.

///// Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$ and note that $T_{[e]\to[e]} = A$. Then

$$T_{[v]\to[v]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Id_{[v]\to[e]} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and $Id_{[e]\to[v]} = (Id_{[v]\to[e]})^{-1}$; since $Id_{[v]\to[e]}$ is orthogonal one has

$$Id_{[e]\to[v]} = Id^T_{[v]\to[e]} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Using that $A = Id_{[v]\to[e]} T_{[v]\to[v]} Id_{[e]\to[v]}$

$$A^{20} = Id_{[v]\to[e]} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{20} Id_{[e]\to[v]} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Aufgabe 4

Betrachte das System von Differentialgleichungen

$$(S) \begin{cases} y_1' &= 3y_1 - 5y_2 + t \\ y_2' &= y_1 - y_2 + 1 \end{cases}.$$

(i) Bestimme die allgemeine reelle Lösung des zugehörigen homogenen Systems.

///// In matrix notation (S) reads

$$y' = Ay, \qquad A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$$

and the general complex solution is $y(t) = e^{At} y_0$. We have $\chi_A = (3-\lambda)(-1-\lambda)+5$ which implies $\lambda_1 = 1-i$, $\lambda_2 = 1+i$. For the eigenvectors (applying Gauss algorithm)

$$\begin{pmatrix} 3-1+i & -5 \\ 1 & -1-1+i \end{pmatrix} \to \begin{pmatrix} 1 & -2+i \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad v_1 + (-2+i) v_2 = 0$$

so $v^{(1)} = (2-i, 1)$ and

$$\begin{pmatrix} 3-1-i & -5 \\ 1 & -1-1-i \end{pmatrix} \to \begin{pmatrix} 1 & -2-i \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad v_1 + (-2-i) v_2 = 0$$

hence $v^{(2)} = (2+i, 1)$. The general complex solution is $y(t) = c_1 e^{\lambda_1 t} v^{(1)} + c_2 e^{\lambda_2 t} v^{(2)}$ with $c_1, c_2 \in \mathbb{C}$. Remark $e^{\alpha + i\beta} = e^{\alpha}(\cos(\beta) + i \sin(\beta))$. The general real solution of the homogeneous system is

$$y_h(t) = a_1 e^t \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + a_2 e^t \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}, \quad a_1, a_2 \in \mathbb{R}$$

/////

(ii) Bestimme eine partikuläre reelle Lösung von (S).

///// We use the following ansatz

$$y_p(t) = \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix}$$

and so we obtain

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3(a_1 t + b_1) - 5(a_2 t + b_2) + t \\ a_1 t + b_1 - a_2 t - b_2 + 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} (a_1 - 3b_1 + 5b_2) + t(-3a_1 + 5a_2) \\ (a_2 - b_1 + b_2) + t(-a_1 + a_2) \end{pmatrix}$$

which implies

$$y_p(t) = \begin{pmatrix} \frac{1}{2} t - \frac{3}{2} \\ \frac{1}{2} t - 1 \end{pmatrix}. \quad /////$$

(iii) Löse das Anfangswertproblem von (S) mit $y_1(0) = -\frac{3}{2}$, $y_2(0) = 0$.

///// We have

$$y = y_h + y_p = a_1 e^t \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + a_2 e^t \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2} t - \frac{3}{2} \\ \frac{1}{2} t - 1 \end{pmatrix}$$

and

$$y_1(0) = 2a_1 + a_2 - 3/2 = -3/2, \quad y_2(0) = a_1 - 1 = 0 \quad \Rightarrow a_1 = 1,\ a_2 = -2. \quad /////$$

Aufgabe 5 Betrachte die folgende Differentialgleichung

$$y'' + y' - 2y = 3e^{-2t}.$$

(i) Bestimme die allgemeine reelle Lösung.

///// *Homogenous equation.* Let us take the Ansatz $e^{\lambda t}$, therefore

$$(\lambda^2+\lambda-2)e^{\lambda t}=0 \quad \Rightarrow \lambda^2+\lambda-2=0 \quad \Rightarrow \lambda_1=-2, \lambda_2=1$$

and so we have

$$y_h(t)=ae^{-2t}+be^t, \quad a,b\in\mathbb{R}$$

Inhomogenous equation. Ansatz

$$y_p(t)=ct\mathrm{e}^{-2t}, \quad c\in\mathbb{R}$$

yields

$$y''+y'-2y=ce^{-2t}(-2+4t-2-2t+1-2t)=ce^{-2t}(-3)$$

hence

$$y_p=-t\mathrm{e}^{-2t}.$$

General real solution.

$$y(t)=ae^{-2t}+be^t-t\mathrm{e}^{-2t}, \quad a,b\in\mathbb{R}. \quad /////$$

(ii) Bestimme die Lösung des Anfangwertproblems mit

$$y(0)=0, \quad y'(0)=2.$$

///// One computes

$$y(0)=a+b=0, \quad y'(0)=-2a+b-1=2$$

so $a=-1$ and $b=1$. /////

(iii) Bestimme die Lösung der Differentialgleichung für welche $y(0)=1$ und $\lim_{t\to\infty} y(t)=0$.

///// One computes

$$y(0)=a+b=1 \quad \Rightarrow \quad y(t)=ae^{-2t}+(1-a)e^t-t\mathrm{e}^{-2t}$$

hence we need $a=1$, $b=0$. /////

Aufgabe 6 Welche der folgenden Aussagen sind wahr und welche sind falsch? (Mit Begründung)

(i) Es existieren Vektoren $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$ in $\mathbb{R}^3$, welche linear unabhängig sind.

///// False. Since $\dim(\mathbb{R}^3)=3$, any four vectors in $\mathbb{R}^3$ have to be linearly dependent. /////

(ii) Der Kegelschnitt $K_f=\{(x_1,x_2)\in\mathbb{R}^2 \mid f(x_1,x_2)=0\}$ mit

$$f(x_1,x_2)=3x_1^2+4x_1x_2+x_2^2-15$$

ist eine Ellipse.

///// False. One can write

$$f(x_1, x_2) = \langle A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rangle - 15 \quad \text{with} \quad A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since $\det(A) = -1 < 0$ the two eigenvalues have different signs, i.e.$\lambda_1 < 0 < \lambda_2$. This implies that K_f is a hyperbola and not an ellipse. /////

(iii) Sei $T : \mathbb{R}^3 \to \mathbb{R}^3$ eine lineare Abbildung. Falls $\{Tv \mid v \in \mathbb{R}^3\} \neq \mathbb{R}^3$, dann ist 0 ein Eigenwert von T.

///// True. Since T is not regular we know that $Tv = 0$ has as a nontrivial solution v, hence 0 is an eigenvalue. /////