Correction Probeprüfung

Aufgabe 1 (a) $\det A = 2\times(-1) - 1\times(-3) = 1$

(b) $\left(\begin{array}{ccc|c} 2 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & -2 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & a \\ 0 & -1/2 & 1/2 & b-1/2a \\ 0 & 1 & -2 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & a \\ 0 & -1/2 & 1/2 & b-1/2a \\ 0 & 0 & -1 & c+2b-a \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 2 & 1 & 1 & a \\ 0 & 1 & -1 & a-2b \\ 0 & 0 & 1 & a-2b-c \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & 1 & a \\ 0 & 1 & 0 & 2a-4b-c \\ 0 & 0 & 1 & a-2b-c \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -a+3b+c \\ 0 & 1 & 0 & 2a-4b-c \\ 0 & 0 & 1 & a-2b-c \end{array}\right)$ and thus $A^{-1} = \begin{pmatrix} -1 & 3 & 1 \\ 2 & -4 & -1 \\ 1 & -2 & -1 \end{pmatrix}$

Aufgabe 2 (a) $\det A = 1\times1 - a\times2 - a^2\times(-1) = a^2 - 2a + 1 = (a-1)^2$

So $v^{(1)}, v^{(2)}, v^{(3)}$ are linearly dependent $\Leftrightarrow \det A = 0 \Leftrightarrow a = 1$.

(b) $\langle v^{(1)}, v^{(2)}\rangle = 1 + a + a^2$; since $\Delta = 1-4 < 0$, $\langle v^{(1)}, v^{(2)}\rangle > 0$ for all $a \in \mathbb{R}$

and there are no solution.

(c) u hat coordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the basis $[v] \Leftrightarrow u = x v^{(1)} + y v^{(2)} + z v^{(3)}$

which yields the system (since $v^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$)

$\begin{cases} x + y + z = 1 \\ y = -1 \\ -y + z = 3 \end{cases}$. Thus $y = -1$, then $z = 2$ and $x = 0$.

Aufgabe 3 (a) We compute the characteristic polynom

$P(z) = \det(A - z\,\mathrm{Id}) = -z^3 + 2z^2 + z - 2.$

We observe that 1 is an obvious root, so $P(z) = (1-z)(z^2 - z - 2)$ and the roots of $z^2 - z - 2$ are -1 and 2. Thus A has three simple (algebraic and geometric) eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$.

(b) A is symmetric, thus there is an orthonormal basis of eigenvectors, and A is diagonal in this basis. We find first an eigenvector with eigenvalue $\lambda_1 = 1$ by solving $\begin{cases} x + y = x \\ x + z = y \\ y + z = z \end{cases}$ we find for instance $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Similarly $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector for the eigenvalue $\lambda_2 = -1$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ an eigenvector for the eigenvalue $\lambda_3 = 2$. We normalize those vectors to get the orthonormal basis $v^{(1)} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$ $v^{(2)} = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$ $v^{(3)} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$.

(c) From (b) and the course, we have

$$S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \quad \text{and} \quad S^{-1} = S^T = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

<u>Aufgabe 4</u> The matrix A is symmetric, so we can apply Sylvester's criterion (Thm 5.3.1 in the script): A is positive definite if and only if the determinants of the sub-matrices $A_1 = 5$ $A_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $A_3 = A$ are all >0.
We find $\det A_1 = 5$, $\det A_2 = 1$, $\det A_3 = 1$, so A is definite positive

<u>Aufgabe 5</u> (a) We first compute the eigenvalues and eigenvectors. The characteristic polynom is $z^2 + 1$, so the two eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.
An eigenvector for the eigenvalue $\lambda_1 = i$ is $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$, and therefore a complex solution is

$$t \mapsto e^{it}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} \cos t + i \sin t \\ \cos t + \sin t + i(\sin t - \cos t) \end{pmatrix}$$

We deduce that the general solution to the system is

$$y_1(t) = a\cos t + b \sin t \qquad a, b \in \mathbb{R}.$$

$$y_2(t) = a(\cos t + \sin t) + b(\sin t - \cos t) = (a-b)\cos t + (b+a)\sin t$$

(b) The conditions $y_1(0) = 1$ and $y_2(0) = 2$ yield the system $\begin{cases} a = 1 \\ a - b = 2 \end{cases}$
that is $a = 1$, $b = -1$.
The solution to the initial value problem is thus

$$y_1(t) = \cos t - \sin t, \quad y_2(t) = 2\cos t$$