Aufgabe 1

(a) $\langle v^{(1)}, v^{(2)} \rangle = a+b+1$, $\langle v^{(1)}, v^{(3)} \rangle = a+c+1$, $\langle v^{(2)}, v^{(3)} \rangle = b+c+1$

Thus we have to solve the system $\begin{cases} a+b+1=0 & (1) \\ a+c+1=0 & (2) \\ b+c+1=0 & (3) \end{cases}$

$(1)-(2) \Rightarrow b=c$. Then in (3) $b=c=-1/2$. Finally $a=-1/2$.

<u>Conclusion</u> $v^{(1)}, v^{(2)}, v^{(3)}$ paarweise orthogonal $\Leftrightarrow a=b=c=-1/2$.

(b) 1. $\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2 \neq 0$. Thus $[v^{(1)}, v^{(2)}, v^{(3)}]$ is a basis.

2. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ with $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

One finds $A^{-1} = \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$, that is

$$\begin{cases} x' = -x/2 + y/2 + z/2 \\ y' = x/2 - y/2 + z/2 \\ z' = x/2 + y/2 - z/2 \end{cases}$$

3. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} \Leftrightarrow \begin{cases} x/2 + y/2 + z/2 = 0 \\ x/2 + y/2 + z/2 = 0 \\ x/2 + y/2 + z/2 = 0 \end{cases}$ Three times the same equation

$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 0$ is the equation of a plane in $\mathbb{R}^3$.

A basis, is, for example $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Aufgabe 2

(a) 1. A symmetric $\iff A = A^T$

2. A orthogonal $\iff A A^T = Id_{\mathbb{R}^d} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

(b) We know that for every matrix $M \in \mathbb{R}^{d \times d}$, $\det M = \det M^T$ and $\det(MN) = \det M \cdot \det N$, $N \in \mathbb{R}^{d \times d}$.

Thus $1 = \det \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \det(A A^T) = \det A \cdot \det A^T = (\det A)^2$

We conclude that $\det A = 1$ or $\det A = -1$.

(c) We know that the eigenvalues of a symmetric matrix are real.
We know that the eigenvalues of an orthogonal matrix are complex numbers with modulus $= 1$.
The only real numbers with modulus $= 1$ are $+1$ and -1.

Aufgabe 3

(a) Characteristic polynom $P(\lambda) = \det(A - \lambda \mathrm{Id})$

$$P(\lambda) = \det \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$\lambda = 1$ is an obvious root of P. $P(\lambda) = (1-\lambda)(\lambda^2 - \lambda - 2)$

The other roots are $\lambda = \frac{1+\sqrt{9}}{2}$ and $\lambda = \frac{1-\sqrt{9}}{2}$, i.e. there are three eigenvalues: $1, -1$, and 2. Each has algebraic and geometric multiplicity 1.

(b) The matrix A is symmetric. Therefore there exists an orthonormal basis of $\mathbb{R}^3$ with eigenvectors of A.

The eigenspace for the eigenvalue 1 is determined by the system
$x - y = x$, $-x + z = y$, $y + z = z$. That is $y = 0$, $x = z$.

$V^{(1)} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ is a unit vector in this eigenspace

The eigenspace with eigenvalue -1 is determined by the system
$x - y = -x$, $-x + z = -y$, $y + z = -z \iff y = 2x$ & $y = -2z$.

$V^{(2)} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$ is a unit vector in this eigenspace

The eigenspace with eigenvalue 2 is determined by the system
$x - y = 2x$, $-x + z = 2y$, $y + z = 2z \iff x = -y$ & $y = z$.

$V^{(3)} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$ is a unit vector in this eigenspace

(3) We can take $S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \end{pmatrix}$

S is orthogonal, thus $S^{-1} = S^T = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$

Aufgabe 4

(a) $A^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and we get

$$B = A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

(b) B is symmetric

$\det B_1 = 1 > 0$

$\det B_2 = \det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = 2 > 0$

$\det B_3 = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix} = 1 > 0$

Thus B is positive definite.

Aufgabe 5

(a) We consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$P(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$

There are two eigenvalues: $\lambda = -1$ and $\lambda = 3$.

An eigenvector for $\lambda = -1$ is (for instance) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

An eigenvector for $\lambda = 3$ is (for instance) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We thus obtain two special solutions $\begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$ and $\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$

and the general solution is

$y_1(t) = -ae^{-t} + be^{3t}$, $y_2(t) = ae^{-t} + be^{3t}$

(b) The initial value problem yields $-a+b = -1$ $a+b = 3$

Thus $b = 1$, $a = 2$ and the solution is

$y_1(t) = -2e^{-t} + e^{3t}$ $y_2(t) = 2e^{-t} + e^{3t}$.